Geometry of Locally Finite Spaces

Presentation of the monograph

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Abstract

The main topic of the monograph [Kov2008] is an axiomatic theory of locally finite topological spaces and the digital geometry based on this theory as well as their applications to computer imagery. Locally finite spaces as compared to classical continuous spaces have the advantage that they can be explicitly represented in a computer. A new set of axioms of digital topology is suggested in the book. Basic topological notions are derived from the axioms; properties of locally finite spaces are investigated. The book contains 32 theorems with proofs. This theory provides a bridge between topology and computer science while being is in full agreement with classical topology. Along with theoretical foundations numerous efficient algorithms for solving topological and geometrical problems are presented in the book. Most algorithms are accompanied by a pseudo-code facilitating their practical employment. The pseudo-code is based on the C++ programming language. The book contains a new approach to digital geometry based exclusively on the theory of locally finite spaces. It is independent of Euclidean geometry. This is an important contribution to the basic research that leads also to numerous new solutions of applied problems. Examples of solutions, the corresponding algorithms and the obtained results are presented in the book. The monograph is a compendium of the results of the author's research in digital topology, digital geometry and computer imagery during the last twenty years. It makes possible to employ these results in computer science, specially in medical and technical image analysis.

1 Introduction

The monograph presents the most important and accomplished results of author's research in the area of digital topology, digital geometry and computer imagery. It is devoted to the theory of locally finite topological spaces and their applications. A locally finite space is a topological space whose each element possesses a neighborhood containing a finite number of elements. Such spaces are in contrast to classical continuous spaces explicitly representable in a computer.

The book presents an axiomatic approach to topology and geometry of locally finite spaces with applications to computer imagery and to other research area. It contains 332 pages, 120 figures with 12 color tables among them, and 85 literature references. There are 32 theorems proved in the book. It also contains numerous algorithms most of which are accompanied by a pseudo-code based on the C++ programming language.

The contents of most important sections of the book are represented in what follows.

2 Locally Finite Topological Spaces

The theory of locally finite spaces serves to overcome the discrepancy between theory and applications existing in geometry and calculus: the traditional way of research consists in making theory in Euclidean space while applications deal only with finite discrete sets. The reason of the latter is that even a small subset of Euclidean space cannon be explicitly represented in a computer because such a subset, no matter how small it is, must contain infinitely many points.

Locally finite spaces are on one hand theoretically consistent and conform with classical topology and on the other hand explicitly representable in a computer.

3 Aims of the Monograph

The author wishes to demonstrate that it is possible to develop a locally finite topology well suited for applications in computer imagery and independent of the topology of the Euclidean space.

The second aim is to present some advises for developing efficient algorithms in computer imagery based on the topology and geometry of locally finite spaces, in particular of abstract cell complexes. Numerous algorithms of that kind are presented in the monograph.

The main topics of the monograph are:

Axiomatic Approach to Digital Topology;
Abstract Cell Complexes – an Important Particular Case;
Continuous Mappings among Locally Finite Spaces;
Digital Lines and Planes;
Theory of Surfaces in a Three-Dimensional Space;
Data Structures;
A Universal Algorithm for Tracing Boundaries in *n*D spaces;
Labeling Connected Components;
Tracing, Encoding and Reconstructing Surfaces in 3D spaces;
Topics for Discussion – Irrational Numbers; Optimal Estimates of Derivatives;
Problems to Be Solved.

4 New Axioms

Why was a new set of axioms suggested?

The relation of axioms of the classical topology to the demands of computer imagery is not clear for a non-topologist. It is e.g. not clear, why do we need the notion of open subsets satisfying classical axioms.

The new axioms are related to the notions of connectedness and to that of the boundary of a subset. These notions are important for applications, in particular for image analysis.

We have demonstrated that classical axioms can be deduced from the new axioms as theorems. In this way classical axioms become related to the desired properties of connectedness and of boundaries.

Axiom 1: For each space element e of the space S there are certain subsets containing e, which are neighborhoods of e. The intersection of two neighborhoods of e is again a neighborhood of e. Each element e has its smallest neighborhood SN(e).

Axiom 2: There are space elements, which have in their SN more than one element.

Axiom 3: The frontier Fr(T, S) of any subset $T \subset S$ is thin.

The notion of a thin frontier is exactly defined in the book. Fig. 1a and 1b illustrate this notion.

In Fig. 1a space elements are squares with the well-known 4-neighborhood relation. The frontier of the shaded area consists of the squares labeled by black and white disks. The frontier is thick. In Fig. 1b space elements are squares, lines and dots. The frontier of the shaded area consists of bold lines, both solid and dotted, and of dots labeled by black and white disks. It is thin.

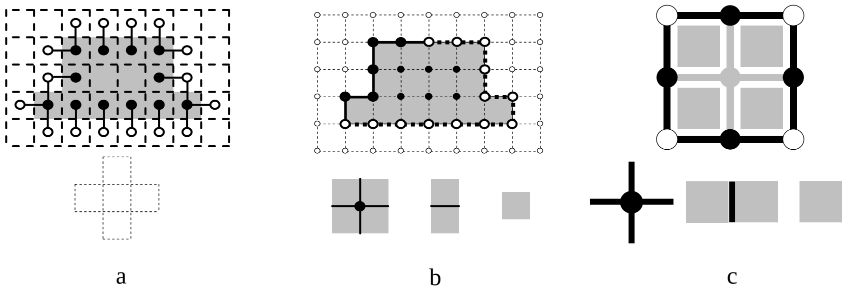

Fig. 1 Examples of frontiers:
A thick frontier (a); a thin frontier (b); a frontier with gaps (c)

Axiom 4: The frontier of Fr(T, S) is the same as Fr(T, S), i.e. Fr(Fr(T, S), S)=Fr(T, S).

Fig. 1c illustrates the case not satisfying Axiom 4. An important property of the frontier is, non-rigorously speaking, that it must have no gaps, which is not the same, as to say that it must be connected. More precisely, this means that the frontier of a frontier F is the same as F. For example, the frontier in Fig. 1c has gaps represented by white disks. Let us explain this. Fig. 1c shows a space S consisting of squares, lines and dots. The neighborhood relation N is in this case non-transitive: The neighborhood SN(L) of a line L contains one or two incident squares, while the neighborhood SN(P) of a dot P contains some lines incident to P *but no squares*. The SN of a square is the square itself. The subset T under consideration is represented by gray elements. Its frontier Fr(T, S) consists of black lines and black dots (disks) since these elements do not belong to T, while their SNs intersect T. The white dots do not belong to F=Fr(T, S) because their SNs do not intersect T. These are the gaps. However, Fr(F, S) contains the white dots because their SNs intersect both F and its complement (at the dots themselves). Thus in this case the frontier F=Fr(T, S) is different from Fr(F, S).

5 Properties of ALF Spaces

We call a locally finite space satisfying our Axioms an ALF space. We have demonstrated in Section 2.3 of the book that the classical axioms can be deduced as theorems from our Axioms and that an ALF space is a *particular case* of the classical T_0 space, but not of a T_1 space.

An abstract cell complex (called AC complex) is a particular case of an ALF space characterized by an additional feature: the dimension function *dim*(a), which assigns a non-negative integer to each space element a in such a way that if $b \in$SN(a), then *dim*(a)≤*dim*(b). Elements of an AC complex are called *cells*. We use the well-known definition of abstract cell complexes suggested by Steinitz [Stein08]:

Definition AC: An *abstract cell complex* (AC complex) $C=(E, B, dim)$ is a set E of abstract elements (cells) provided with an asymmetric, irreflexive, and transitive binary relation $B \subset E \times E$ called the *bounding relation*, and with a dimension function $dim: E \rightarrow I$ from E into the set I of non-negative integers such that $dim(e') < dim(e'')$ for all pairs $(e', e'') \in B$.

A cell is never a subset of another cell. It is usual to write $e'<e''$ for $(e', e'') \in B$.

We have augmented the above definition by a topological definition of the dimensions of space elements. Dimensions of cells represent the partial order corresponding to the bounding relation. Let us call the sequence $a<b< ... <k$ of cells of a complex C, in which each cell bounds the next one, a *bounding path* from a to k in C. The number of cells in the sequence *minus one* is called the *length* of the bounding path.

Definition DC (dimension of a cell): The dimension $dim(c, C)$ of a cell c of a complex C is the length of the *longest bounding path* from any element of C to c.

This definition is in correspondence with the well-known notion of the topological dimension or height of an element of a partially ordered set [Birk61].

According to Definition DC the dimension of a cell c is defined *relative* to a subcomplex containing the cell c because the length of the longest bounding path can be different in different subcomplexes.

An example of calculating the dimensions of cells is shown in Fig. 2. The cell v has dimension 3 since the length of the path $p<e<f<v$ is equal to 3.

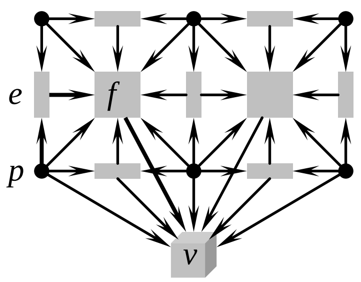

Fig. 2 A complex with bounding relations represented by arrows.
An arrow points from a to b if a bounds b

The dimension of the space elements is an important property. Using dimensions prevents one from errors which can occur when using an LF space without dimensions. An example of a typical error is presented in the book.

We have introduced the notion of an n-dimensional Cartesian complex as the Cartesian product of n one-dimensional complexes [Kov86]. This gives us the possibility to define coordinates of the cells. We call them *combinatorial coordinates*.

Fig. 3 shows the closures and the smallest neighborhoods (SONs) of cells of Cartesian complexes of different dimensions.

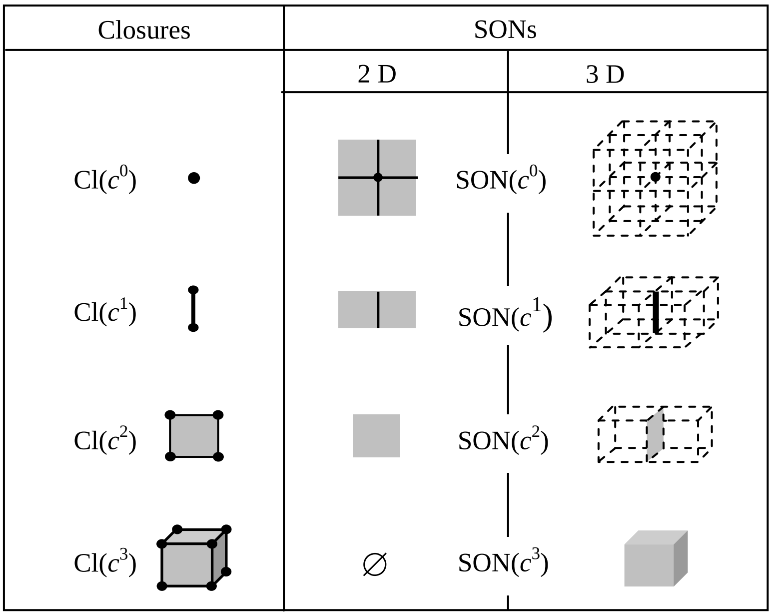

Fig. 3 Closures and SONs of cells of Cartesian AC complexes

6 Combinatorial Homeomorphism, Balls and Spheres

The notion of the homeomorphism of two sets is a fundamental notion of topology: Two sets are called homeomorphic or topologically equivalent if there is a continuous map from one to the other whose inverse is also continuous. There is another classical way to define the homeomorphism. It is directly applicable to complexes and can be extended to other locally finite spaces. It is called the *combinatorial homeomorphism* and is based on the notion of *elementary subdivisions* of cells [Stil95, p. 24]. We shall apply it to AC complexes.

The original concept of an AC complex is too general: It is e.g. possible to define a "strange" AC complex with a one-dimensional cell (1-cell) bounded by more than two 0-cells, or with a 2-cell that has a hole, or with a 3-cell being a torus, etc. All this does not contradict the above Definition AC. To avoid such situations elementary subdivisions have been defined in classical topology on the base of the Euclidean topology and Euclidean complexes (see e.g. a modern survey in [Stil95]). Since our aim is to develop a theory independent of the theory of Euclidean spaces, we shall suggest new definitions based exclusively on the topology of AC complexes. We suppose that the notion of the combinatorial homeomorphism is not applicable to any complex. There must be a limitation excluding "strange" complexes as mentioned above. This limitation should be of the same nature as the classical limitation defining Euclidean cells as convex sets.

One possible way is to try to introduce a class of complexes which are in certain sense similar to Cartesian ones since Cartesian complexes have the desired properties: A 1-cell is bounded by no more than two 0-cells; a 2-cell has no holes; a 3-cell is a topological three-dimensional ball, etc. However, we do not see a possibility to define the class of complexes homeomorphic to Cartesian ones before having defined the notions of a topological ball and a topological sphere, which are necessary to define the subdivisions. Our intention is to define topological notions *before and independently of* the geometric ones, because we believe that geometry can be consistently constructed only after the corresponding topological space is already defined. Topology must be the foundation of geometry and not vice versa. Therefore we do not employ geometric notions like metric and Euclidean coordinates in topological definitions. Thus we cannot employ the classical definition of a topological ball, which is a set of points having a limited *distance* to a center point.

We have accommodated the notions of a topological ball and of a topological sphere to complexes, while introducing the notions of *combinatorial balls and spheres*. To avoid the usage of "strange" complexes we have introduced the notions of *proper* cells and complexes which can be regarded as a substitution for convex cells of Euclidean complexes.

The notion of a proper cell has lead to new definitions of combinatorial balls and spheres independent of geometry and metric. The following Fig. 4 is an illustration to these notions.

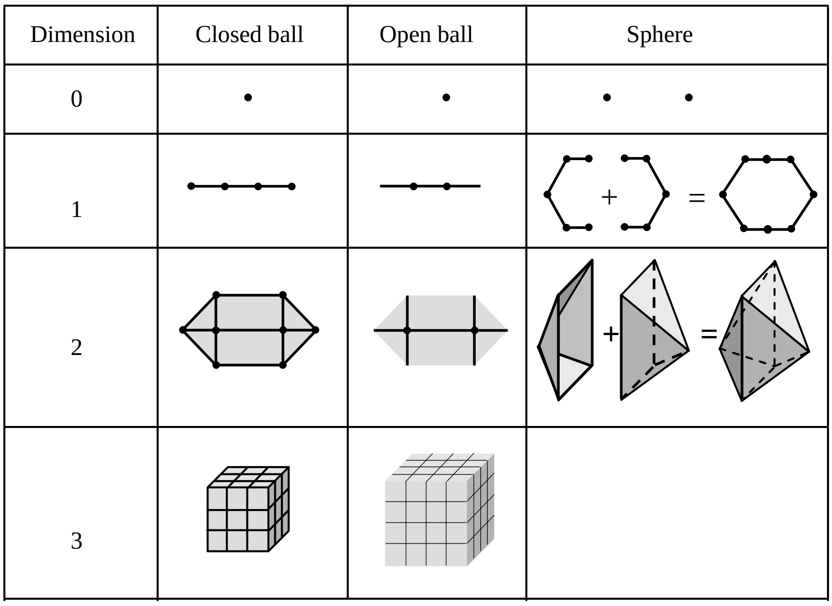

Fig. 4 Examples of AC balls and spheres of dimensions from 0 to 3

The purely combinatorial definitions of a combinatorial ball and a combinatorial sphere independently of the Euclidean space have provided the possibility to justify the well-known notion of combinatorial homeomorphism for locally finite spaces with no reference to Euclidean space. It is based on elementary subdivisions of cells. Fig. 5 shows an example of the elementary subdivision of a two-dimensional cell.

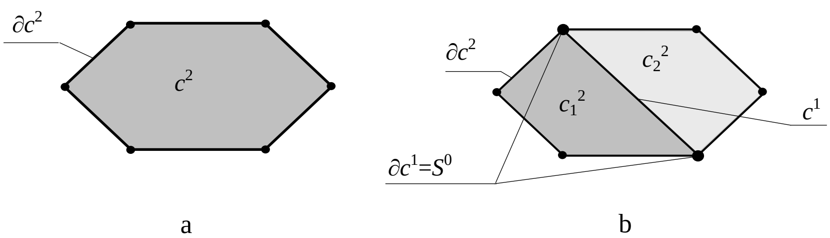

Fig. 5 An example of the elementary subdivision of a 2-cell; the original cell (a) and its subdivision (b)

Exact definitions and proofs are to be found in the book.

Fig. 6 presents an example demonstrating the combinatorial homeomorphism of a square and a triangle.

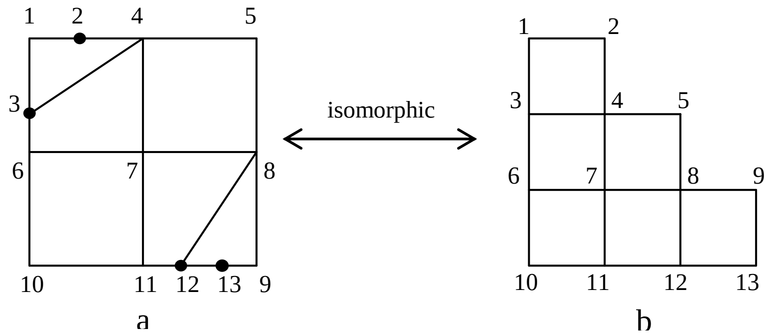

Fig. 6 A subdivision of a digitized square (a) which is isomorphic to a digitized triangle (b); "fat points" denote the new points introduced during the subdivision

We also have generalized the notions of a boundary and a frontier while having introduced the notions of an *opening frontier, opening boundary and generalized boundary*. Thus for example a punctured two-dimensional sphere (i.e. a sphere without a point) is according to the new definitions a manifold with an opening boundary while from the classical point of view it is no manifold at all.

7 Continuous Functions and Connectedness Preserving Maps

In classical topology homeomorphism is defined by means of continuous mappings between topological spaces. The possibilities to apply this idea to locally finite spaces are rather limited: We have demonstrated in Section 4.2 that isomorphism is the *only classical homeomorphism possible* between two locally finite spaces (LFS)

We have demonstrated that it is impossible by classical means to continuously map one LFS onto a "greater" space, i.e. onto a space containing more elements. We have suggested considering more general correspondences between the spaces *X* and *Y*, assigning to each cell of *X* a *subset of Y* rather than a single cell [Kov93, Kov94]. A correspondence is called a *connectedness preserving* one (CPM) if it maps each connected subset of *X* to a connected subset of *Y*. In agreement with the classical definition of “continuous”, a connectedness preserving map is continuous if the preimage of each open subset is open.

Consider the CPM F: $X \rightarrow Y$, where X and Y are complexes; the subcomplex $F(x)$ which is the image of $x \in X$ and the subcomplex $F^{-1}(y)$ which is the preimage of $y \in Y$. Let us denote by $V(x, y)$ the connected component of $F(x)$ containing y and by $H(x, y)$ the connected component of $F^{-1}(y)$ containing x. A correspondence F is called *simple* if for each pair $(x, y) \in F$ at most one of the sets $V(x, y)$ and $H(x, y)$ contains more than one element. Fig. 7 shows two examples of CPMs.

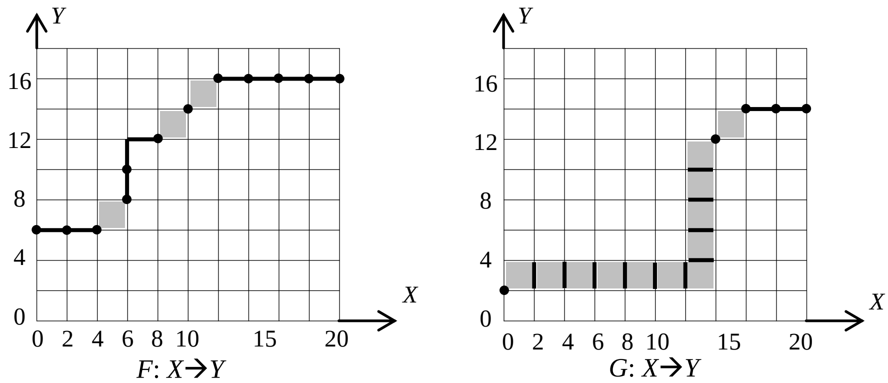

Fig. 7 Examples of CPMs: *F* is simple, but not continuous; *G* is continuous, but not simple.

We have also demonstrated that combinatorial homeomorphism $X \sim Y$ according to the Definition CH (Section 3.6, p. 57) uniquely specifies a continuous CPM F: $X \rightarrow Y$ whose inverse correspondence is also a continuous CPM.

8 Digital Geometry. Digital Lines and Planes

Section 6 of the book presents the new concept of digital geometry, in particular an introduction to digital analytical geometry. Section 7 presents the theory of digital lines and planes. It contains definitions of lines and planes which are (in contrast to all definitions known to the author) independent of the corresponding Euclidean notions. This means that a digital line is defined not as the result of digitizing a Euclidean line. We first define a half-plane as a set of elements of a 2D Cartesian complex whose combinatorial coordinates satisfy a linear inequality and then a digital straight segment as a connected subset of the frontier of a half-plane.

We distinguish between two types of digital curves in a two-dimensional space: visual curves are sequences of pixels (two-dimensional cells) and are well suited for representing curves in an image; boundary curves are sequences of zero- and one-dimensional cells and are well suited for purposes of image analysis. We consider mainly boundary curves rather than visual curves.

Section 7 of the book presents a complete theory of digital straight segments (DSS) being regarded as boundary curves. Equation defining such a DSS and the algorithm of recognizing a DSS are similar to the well-known equations and algorithms for visual lines, but there are also some important differences.

A fast algorithm for subdividing a digital boundary curve into longest DSS is presented in this section. A method of economically and loss-free encoding sequences of DSSs is described in Section 7.2.5. “Loss-free” means that a segmented digital image can be exactly reconstructed from the code of the boundaries of the segments.

Section 7.3 contains the theory of digital planes. They are considered, similarly to the DSSs, as boundaries of half-spaces. They consist not of voxels but rather of cells of dimensions 0 to 2. Section 8 is devoted to the theory of surfaces in a three-dimensional space and Section 9 to the theory of digital arcs.

9 Applications of the DSSs

The following applications of the DSSs are described in the book:

1. Estimating the perimeter of a subset of a two-dimensional space.
2. Representing objects in 2D images as polygons for the purpose of shape analysis.
3. Economical and exact encoding of images.

Let us explain the latter application. There exist many different DSSs going through two given points. Three additional integer parameters L, M, N must be specified to distinguish them: M/N is the slope of the base of the DSS, L is the value of the left side of the equation $H(x, y)=0$ of the base at the starting point of the DSS. A DSS can be *exactly* reconstructed from the coordinates of its end points and the additional parameters. There is a possibility to economically encode a sequence of DSSs by means of these data while using at the average 2.3 byte per DSS.

Fig. 8 presents an example of economical encoding of an image by DSS polygons and of fast recognition of all partially injured disk-shaped objects.

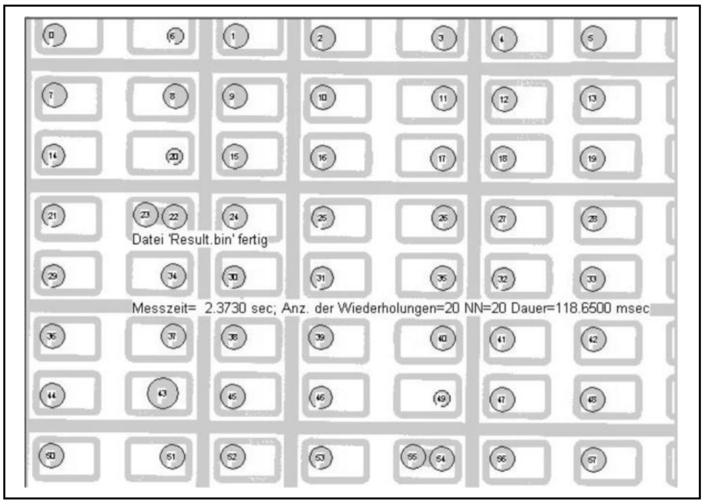

Fig. 8 Example of an image of a wafer with recognized disk-shaped objects

The binary image of 832×654 pixels shown in Fig. 8 was encoded by DSSs. All partially injured disk-shaped objects in the image were correctly recognized. The whole processing of encoding the image and recognizing 57 objects took 20 ms on a PC with a Pentium processor of 700 MHz

11 Other Applications and Algorithms

Section 11 of the book starts with *recommendations for designers of algorithms*. We recommend not to use adjacency relations (4- and 8-adjacency in the 2D case; 6-, 18- and 26-adjacency in the 3D case), but rather to consider all topological and geometrical problems from the point of view of locally finite topological spaces (ALF spaces) or, even better, of AC complexes. Complexes have some advantages as compared to other ALF spaces due to the presence of the dimensions of cells. The dimensions of cells make the work with a topological space easier and more illustrative and help avoiding contradictions. Section 11.1 contains concrete recommendations for ways of using AC complexes for the development of algorithms in computer imagery.

Section 11.2 contains descriptions of various algorithms for tracing and encoding boundaries in 2D images and in 2D subspaces of n-dimensional spaces. Among them is the *universal* algorithm for tracing boundaries of 2D slices in nD images; n=2, 3, 4. This algorithm can be used *without any changes* in spaces of any dimensions. It has been successfully tested in spaces of dimensions up to 4. It was used for tracing and encoding surfaces in the 3D space.

Section 11.2.6.2 contains the algorithm for *automatically* generating the *block cell list* of a segmented multicolored image. The block cell list is a data structure developed by the author [Kov89]. It enables an economical and loss-free encoding of the image and is well suited for the image analysis because it contains the full topological and geometrical information. Relations between subsets of the image such as incidence, adjacency, inclusion etc. can be extracted from the list without a search. There are in the book some examples of color images exactly reconstructed from the block cell list.

Sections 11.3 to 11.7 of the book contain descriptions of the following algorithms:

1) Loss-free encoding of digital straight segment with additional parameters;
2) Exact reconstruction of n-dimensional images from boundary codes; n=2, 3, 4; three different algorithms;

3) Fast labeling of connected components, two different algorithms;

4) Parallel computing of skeletons of subsets in 2D;

5) Algorithms for topological investigations.

Section 12 describes the method of constructing convex hulls in three dimensional spaces.

Section 13 is devoted to tracing and encoding of surfaces in a three dimensional space. It contains descriptions of the following four algorithms:

a) The simplest encoding of a surface by the depth-first search;

b) Loss-free encoding of a surface by a single Euler circuit of the neighborhood graph of the facets (2-cells);

c) Spiral tracing method automatically producing a handle decomposition of the surface along with its code;

d) Economical encoding of surfaces by the "Hoop Code" with less than 2 bits per facet (two-dimensional cell).

Most algorithms are accompanied by a pseudo-code based on the C++ programming language.

12 Topics for Discussions and Problems to be Solved

The last Section 14 of the monograph is devoted to disputable questions of the necessity and possibility to avoid the usage of irrational numbers and of the optimal method of calculating derivatives of functions that are defined with a limited precision.

We have demonstrated that there is the possibility to develop the theory of computations while using neither real numbers nor Euclidean spaces. Algorithms presented in the book employ this possibility while nevertheless considering topological notions of connectedness and of boundary, which are in accordance with classical topology.

It is desirable to eschew real numbers, because it is impossible to directly implement in practice results theoretically obtained for real numbers and for numerous mathematical notions based on real numbers. There exists no arithmetic of real numbers. This is really the case: for example, the real number $\sqrt{2}+\sqrt{3}$ has no name because this expression is a *problem statement* rather than a name of the result. In the case of rational numbers the result of any arithmetic operation (except division by 0) has a name looking as “numerator/denominator” or as a decimal brake. However, only real numbers of a countable subset have names like π or e. The set of real numbers having no names is therefore uncountable. No arithmetic can exist for objects without names. When performing the calculation corresponding to $\sqrt{2}+\sqrt{3}$ one replaces the summands and the sum by rational numbers approximating them. It is not correct to say that we can calculate a real number with an error which is *as small as desired*. It is e.g. impossible to calculate it with an error less than 10^{-n} with n=10^{100} since there is neither enough time nor memory space for the result (the result would be a number with n=10^{100} digits).

The way out of these difficulties consists in using a locally finite space as the number axis and the above mentioned CPMs (connectedness preserving maps) instead of continuous functions, whose application to locally finite spaces is rigorously limited.

Section 14 also demonstrates that the classical notion of the derivative of a function cannot be applied to maps between locally finite spaces. The definition of the classical derivative is based on the supposition that one is able to compute the values of the function with an arbitrarily high precision. We have demonstrated that, when approximating in a computer the value of the derivative by the well-known expression

$$(f(x+\Delta x)-f(x))/\Delta x$$

with decreasing values of Δx, the results become imprecise at small values of Δx and then even meaningless when the value of Δx becomes comparable with the imprecision of the values of $f(x)$. There is an optimal vale of Δx depending on that imprecision and on the values of higher differences (derivatives) of $f(x)$. The above expression with the optimal value of Δx must be used instead of the classical derivative whereas the classical derivatives may serve only as *approximations* which are the more precise the higher the precision of the values of the function and the lower the absolute values of higher derivatives.

As a demonstration of the above ideas, Section 14.2 contains the inference of the well-known Taylor formula for finite differences.

The contents of most sections of the monograph are supplemented by suggestions of problems to be solved.

13 Conclusion

As far as it is known, presents the monograph the first attempt to develop an axiomatic theory of locally finite spaces and a concept of digital geometry independent of Hausdorff topology and Euclidean geometry. The monograph also indicates ways of applying the obtained theoretical results to computer imagery and other research areas.

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